Weak Identification Robust Methods for Production Function Estimation

Jorge de la Cal Medina

University of Amsterdam, Tinbergen Institute

October 2, 2025

Introduction

- **Production function (PF) estimation:**

$$y = x'\beta + \omega + \varepsilon$$

  Firms choose inputs x after observing state ω (Marschak and Andrews, 1944).
- **Control function approach (CFA):** Proxying ω with a control function allows to estimate β with GMM (Ackerberg et al., 2015).
- **Weak identification:** When proxies have low explanatory power (measurement error) Jacobian near-singular, leading to biased, non-normal estimates and invalid standard inference.

This Paper

Contributions

1. Identification with noisy proxy feasible (under restrictions)
2. Weak proxies (small signal/noise) can cause biased estimates with non-normal distributions.
3. Adapt weak identification-robust methods: bootstrap pre-tests for proxy strength (Angelini et al., 2024) and ID-robust confidence sets (Stock and Wright, 2000).

Related Literature

Production-function identification

- Olley and Pakes (1996) (OP), Levinsohn and Petrin (2003) (LP), Ackerberg et al. (2015) (ACF) discuss identification using (strong) proxies; Gandhi et al. (2020) (GNR) discuss non-identification (in nonparametric models)
- *We consider sources for weak identification*

Weak identification in structural models

- DSGE (Canova & Sala 2009), NK Phillips curve (Mavroeidis, Plagborg-Møller & Stock 2013), BLP demand (Armstrong 2016)
- *We extend this list with structural production-function estimation*

Empirical work

- Empirical applications using control function estimator
  - Trade: Pavcnik (2002); Topalova (2010); Fernandes (2007)
  - Market power: De Loecker et al. (2020); Autor et al. (2020)
  - Finance: Gopinath et al. (2017); Gourinchas et al. (2020)
- *We give guidance on model specification and add to toolbox*

Structure of Presentation

- Identification Analysis
  - Present identification strategy of CFA
  - Weak proxies
- Methods
  - Bootstrap pretest
  - AR–GMM
- Monte Carlo simulation
- Empirical Analysis
- Conclusion

Outline

The Control Function Approach

- General PF specification for firm i at period t:

$$y_{it} = x_{it}'\beta + \omega_{it} + \varepsilon_{it}, \quad \mathbb{E}[\varepsilon_{it}|\mathcal{I}_{it}] = 0 \tag{1}$$

- Markov process for state variable

$$\omega_{it} = \rho\,\omega_{it-1} + \xi_{it}, \quad \mathbb{E}[\xi_{it} \mid \mathcal{I}_{it-1}] = 0 \tag{2}$$

- Demand function h of intermediate inputs m

$$m_{it} = h(\omega_{it}, x_{it}) + \nu_{it} \tag{3}$$

  - strictly monotonic in ω_{it}
  - allow for unobservable ν_{it} (unlike OP, LP, ACF)
- Control function

$$\begin{aligned}\omega_{it} &= h^{-1}(m_{it} - \nu_{it}, x_{it}) \\ &= \Psi(m_{it}, x_{it}) + \eta_{it}, \quad \mathbb{E}[\eta_{it}|x_{it}, m_{it}] = 0\end{aligned}$$

  - $\Psi(m, x) = E[\omega_{it} \mid m_{it} = m,\ x_{it} = x]$

Identification and Estimation

- Proxy equation (first stage)

$$y_{it} = \underbrace{x'_{it}\beta + \Psi(m_{it}, x_{it})}_{\Phi(m_{it}, x_{it})} + \eta_{it} + \varepsilon_{it}$$

- Structural equation (second stage)

$$y_{it} = x'_{it}\beta + \rho\left(\Phi(m_{it-1}, x_{it-1}) - x'_{it-1}\beta\right) - \rho\eta_{it-1} + \xi_{it} + \varepsilon_{it}$$

- Combining both stages allows estimation in one go

$$\begin{aligned} y_{it} &= x'_{it}\beta + \rho\left(y_{it-1} - x'_{it-1}\beta\right) - \rho\varepsilon_{it-1} + \xi_{it} + \varepsilon_{it} \\ \mathbb{E}[\xi_{it} &+ \varepsilon_{it} - \rho\varepsilon_{it-1} | \mathcal{I}_{it-1}] = 0 \\ z_{it} &= \{x_{it-1}, m_{it-1,\dots}\} \in \mathcal{I}_{it-1} \end{aligned}$$

- Straightforward to get CUE (Hansen et al., 1996) for ID-robust inference methods

Nonidentification & Weak Proxies

Identification in GMM

- Moment conditions of second stage

$$\mathbb{E}[f_{it}(\beta_0,\rho_0)] = \mathbb{E}\Big[z_{it}\big(y_{it} - x_{it}'\beta - \rho_0\left(\Phi_{it-1} - x_{it-1}'\beta_0\right)\big)\Big] = 0$$

- Identification requires full-rank Jacobian

$$\mathcal{J} := \left(\mathcal{J}_\beta \,\vdots\, \mathcal{J}_\rho\right) = \left(\mathbb{E}\big[z_{it}(x_{it}' - \rho_0 x_{it-1}')\big] \,\vdots\, \mathbb{E}\big[z_{it}\underbrace{(\Phi_{it-1} - x_{it-1}'\beta_0)}_{\Psi_{it-1}}\big]\right)$$

Nonidentification

- Assume linear control function $\Psi_{it-1} = \gamma_m m_{it-1} + x_{it-1}'\gamma_x$
- $\mathcal{J}$ is rank-deficient, if
  - x_{it} and x_{it-1} collinear
  - $\gamma_m \to 0$

Weak proxies

- Small γ_m (weak proxies) lead to near rank-deficiency $\Rightarrow$ Weak identification
- Example: classical measurement error in proxies attenuates γ_m

Strong ID: Standard Asymptotics

Standard Setup (Hansen, 1982)

- Parameter vector $\theta = (\beta^\top, \rho)^\top$
- Sample moments

$$f_n(\theta) = \frac{1}{n(T-1)} \sum_{i=1}^{n} \sum_{t=2}^{T} f_{it}(\theta)$$

- Estimator and Objective

$$\hat{\theta}_n = \arg\min_{\theta \in \Theta} Q_n(\theta), \quad Q_n(\theta) = n\, f_n(\theta)^\top W_n f_n(\theta)$$

- Under standard Assumptions, if Jacobian *full rank*
  - Consistency: $\hat{\theta}_n \xrightarrow{p} \theta_0$
  - Asymptotic normality: $\sqrt{n}\,(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0,\ V_\theta)$

Weak ID: Nonstandard Asymptotics

Partially identified Setup (Stock and Wright, 2000)

- Nearly singular Jacobian in β:

$$\mathcal{J}_\beta(\theta_0) = \Pi_n = \frac{C}{\sqrt{n}}, \quad C \text{ full column rank,}$$

  with columns of C linearly independent of $\mathcal{J}_\rho(\theta_0) = r \neq 0$.
- $\hat{\rho}_n$ is $\sqrt{n}$-consistent
- Joint CLT

$$\sqrt{n}\begin{pmatrix} f_n(\theta_0) \\ \hat{\rho}_n - \rho_0 \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \psi_f \\ \psi_\rho \end{pmatrix}$$

Limit process of sample moments

$$\sqrt{n} f_n(\beta, \hat{\rho}_n) \xrightarrow{d} \psi_f + C(\beta - \beta_0) + r\psi_\rho + O_p(1)$$

Implications

- $Q_n(\beta, \hat{\rho}_n) = O_p(1)$ for $\beta \neq \beta_0$
- No $\sqrt{n}$–consistency and asymptotic non-normality for $\hat{\beta}_n$

Outline

Bootstrap pre-test for proxy relevance (Angelini et al., 2024)

- H_0: Proxies (and instruments) are strong
- Denote full-sample estimate $\hat{\theta}_n$ with variance V_θ, and BS estimate $\hat{\theta}^*_n$
- Statistic: $\Gamma^*_n = \sqrt{n}\,\hat{V}_\theta^{-1/2}(\hat{\theta}^*_n - \hat{\theta}_n)$
- Asymptotically Gaussian *only* under strong proxies:

$$\Gamma^*_n \xrightarrow{d}_{H_0} \mathcal{N}(0, I)$$

- Testing Procedure
  - Draw BS samples and compute $\Gamma^*_{n,b}$ for each
  - Test (multivariate) normality of $\{\Gamma^*_{n,b}\}_{b=1}^B$
  - If reject normality: treat proxies/instruments as weak
- Key features
  - Post-test inference unaffected
  - Simplicity: only requires $\hat{\theta}^*_n$ and $\hat{V}_\theta$ (straightforward with CUE)

AR-GMM (Stock and Wright, 2000)

- $H_0 : \theta_0 = \theta^*$
- Statistic: $S(\theta^*) = nf_n(\theta^*)'\hat{V}_{ff}(\theta^*)^{-1}f_n(\theta^*)$
- Asymptotically chi-squared

$$S(\theta^*) \xrightarrow{d}_{H_0} \chi^2_{d_\theta}$$

- Confidence set by inversion

$$CS(\alpha) = \{\theta^* : S(\theta^*) \leq \chi^2_{d_f}(1-\alpha)\} \tag{4}$$

- Key features
  - Confidence sets remain valid even under weak identification
  - Not most powerful, since also tests against overidentifying moment conditions
  - More powerful robust tests exist, e.g. Kleibergen (2002), Moreira (2003)
  - Easily implementable once we have CUE

Subset-inference (Kleibergen and Mavroeidis, 2009)

- Parameter partition: $\theta = (\theta_1, \theta_2)$
- Subset null hypothesis $H_0: \ \theta_1 = \theta_1^*$
- Restricted CUE

$$\hat{\theta}_2(\theta_1^*) := \arg\min_{\theta_2} S(\theta_1, \theta_2)\big|_{\theta_1 = \theta_1^*}$$

- Asymptotics: Limiting distribution is stochastically dominated by chi-squared (equality when θ_2 is well identified)

$$S(\theta_1^*, \hat{\theta}_2(\theta_1^*)) \preceq^d_{H_0} \chi^2_{d_f - d_{\theta_2}}$$

- Confidence set

$$CS(\alpha) = \{\theta_1 : S(\theta_1, \hat{\theta}_2(\theta_1)) \leq \chi^2_{d_f - d_{\theta_2}}(1-\alpha)\}.$$

Outline

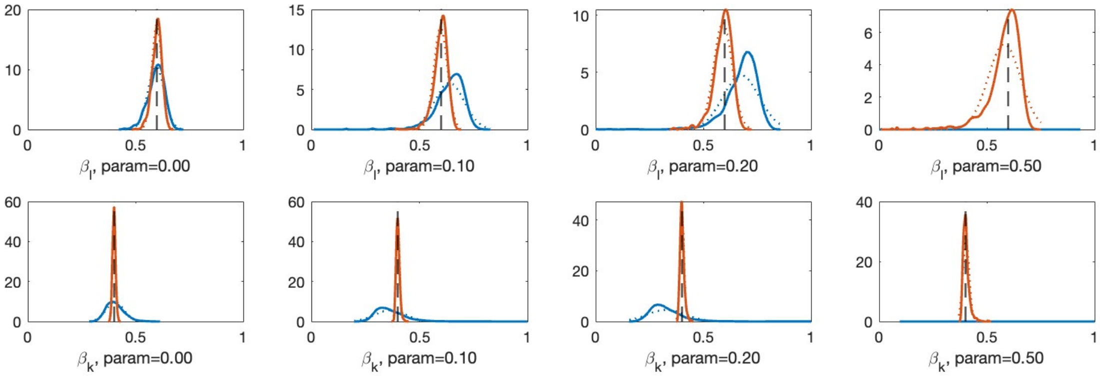
Monte Carlo

- Dynamic model of firm investment and production (ACF)
  - Two inputs capital k and labor l
  - Intermediate inputs, m, subject to measurement error
  - Tweak: reduce optimization error in l from 10% to 5%
- Estimators
  - ACF estimator
  - CUE

Table 1: Bias of simulated estimates (with standard errors). Measurement error in terms of additional % variance of m.

	ACF		CUE	
Meas.	β_l	β_k	β_l	β_k
0.0	−0.005 (0.038)	+0.005 (0.041)	−0.001 (0.023)	0.000 (0.007)
0.1	−0.500 (0.300)	−0.301 (0.299)	+0.038 (0.068)	−0.035 (0.070)
0.2	−0.445 (0.985)	−0.493 (0.983)	−0.461 (0.346)	−0.300 (0.300)
0.5	—	—	−0.532 (1.014)	−0.436 (0.968)

Monte Carlo Estimates of PF Parameters



1000 replications. ACF in blue. CUE in red. Empirical densities: solid. Normal pdf: dotted. Panels ordered left to right for larger measurement-error magnitudes.

- Measurement error ↑: stronger bias and non-normality

Monte Carlo: Inference on PF Parameters

Table 2: Rejection frequencies at 5 % nominal level

Meas.	Joint test (non-robust)[1]	Joint test (robust)[2]	BS joint normality[3]
0.0	0.064	0.060	0.057
0.1	0.059	0.048	0.068
0.2	0.075	0.062	0.167
0.5	0.128	0.050	0.431

Based on 1000 replications. [1]Wald. [2]AR-GMM. [3]Doornik and Hansen (2008).

- Measurement error ↑:
  - Wald over-rejects
  - AR-GMM keeps size
  - BS normality rejections ↑

Outline

Data & empirical setting (Chile & U.S.)

- Replicate Raval (2023) and apply CUE estimator and identification-robust methods
- Chile (ENIA, plant-level)
  - Fabricated metal products (ISIC 381)
  - years 1979–1996
  - plant-year obs. $\approx$ 4,000
  - Materials: plant-reported intermediate consumption (*high precision*)
- U.S. (Compustat, firm-level)
  - Manufacturing (mostly durables incl. metal products; NAICS 33)
  - years 1970–2010
  - firm-year obs. $\approx$ 8,000
  - Materials: proxy via COGS $-$ XLR; mixes labor/overhead (*lower precision*)
- Model & estimators.
  - Cobb–Douglas in k, l, m
  - Control function: third-order polynomial in k, l, m
  - Estimation: ACF baseline + CUE

ENIA, Chile (ISIC 381)

Table 3: Production function estimates (Chile, ISIC 381)

	Estimates		95% CI		BS Normality[3]
Parameter	ACF	CUE	Nonrobust[1]	Robust[2]	p^3
β_k	0.064	0.047	[0.014, 0.080]	[0.003, 0.087]	0.0
β_l	0.122	0.053	[−0.090, 0.195]	[−0.083, 0.185]	0.0
β_m	0.875	0.956	[0.834, 1.078]	[0.848, 1.072]	0.0
Returns to scale	1.060	1.060	—		—

[1]Wald. [2]Subset AR-GMM. [3]Shapiro–Wilk Normality test.

- ACF and CUE estimates in ballpark
- Non-robust and robust CI similar and rel. tight
- Normality is rejected still

Compustat, U.S. (NAICS 33)

Table 4: Production function estimates (U.S., NAICS 33)

	Estimates		95% CI		BS Normality
Parameter	ACF	CUE	Nonrobust[1]	Robust[2]	p[3]
β_k	0.422	0.187	$[-0.660, 1.034]$	$[-0.153, 0.507]$	0.0
β_l	0.411	0.333	$[-1.135, 1.801]$	$(-\infty, 0.787]$	0.0
β_m	0.237	0.219	$[0.106, 0.332]$	$[0.052, 0.332]$	0.0
Returns to scale	1.070	0.739	—		—

[1]Wald. [2]Subset AR-GMM. [3]Shapiro–Wilk Normality test.

- CUE aligns less with ACF
- robust CIs are large, unbounded for β_l (weak ID)
- Normality again rejected

Conclusion

Findings

- Proxies with poor explanatory power (weak proxies) are source of weak identification
- Monte Carlo shows potential large imprecision and non-normality in case of weak proxies
- Robust methods ensure valid inference

Outlook

- Kitchen-sink approach for proxy
- Pretest that does not rely on BS (Andrews, 2018)
- More powerful robust inference (Kleibergen, 2005; Moreira, 2003)
- Other empirical applications

References

Ackerberg, D. A., Caves, K., and Frazer, G. (2015). Identification properties of recent production function estimators. *Econometrica*, 83(6):2411–2451.

Andrews, I. (2018). Valid two-step identification-robust confidence sets for gmm. *The Review of Economics and Statistics*, 100(2):337–348.

Angelini, G., Cavaliere, G., and Fanelli, L. (2024). An identification and testing strategy for proxy-svars with weak proxies. *Journal of Econometrics*, 238(2):105604.

Autor, D., Dorn, D., Katz, L. F., Patterson, C., and Van Reenen, J. (2020). The Fall of the Labor Share and the Rise of Superstar Firms. *The Quarterly Journal of Economics*, 135(2):645–709.

De Loecker, J., Eeckhout, J., and Unger, G. (2020). The rise of market power and the macroeconomic implications. *The Quarterly Journal of Economics*, 135(2).

Doornik, J. A. and Hansen, H. (2008). An omnibus test for univariate and multivariate normality. *Oxford Bulletin of Economics and Statistics*, 70(5):927–939.

Gandhi, A., Navarro, S., and Rivers, D. A. (2020). On the identification of gross output production functions. *Journal of Political Economy*, 128(8). Electronically published.

Gopinath, G., Kalemli-Özcan, , Karabarbounis, L., and Villegas-Sanchez, C. (2017). Capital Allocation and Productivity in South Europe. *The Quarterly Journal of Economics*, 132(4):1915–1967.

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators.